Scattered Hermite Interpolation Using Radial Basis Functions

Xingping Sun
Department of Mathematics
Southwest Missouri State University
Springfield, Missouri 65804

Submitted by Michael Neumann

ABSTRACT

We study the scattered Hermite interpolation problem and find several classes of radial basis functions, including the multiquadrics, which may be implemented for this interpolation scheme.

1. INTRODUCTION

Let $\mathscr{N}$ denote a set of n distinct points in $\mathbb{R}^d$ designated by $x_1, \dots, x_n$. These points are called *nodes*. The basic problem of multivariate interpolation is as follows. A *data function* $\Delta : \mathscr{N} \mapsto \mathbb{C}$ is given, and we seek a function $f : \mathbb{R}^d \mapsto \mathbb{C}$ such that $f | \mathscr{N} = \Delta$. Such a function f is said to *interpolate* Δ.

If the set of nodes has no special structure capable of being exploited, then this problem is called scattered data interpolation. This interpolation scheme is frequently demanded in various data-fitting problems, and many methods have been proposed and discussed in the literature; see the survey articles of Schumaker [15] and Franke [6, 7]. One method that has been used successfully employs radial basis functions. In this case, one seeks an interpolant from the linear space generated by the n functions $\phi(|\cdot - x_1|), \dots, \phi(|\cdot - x_n|)$, where $|\cdot|$ denotes the Euclidean norm and ϕ is a fixed function from $\mathbb{R}^+$ to $\mathbb{C}$. In particular, the function

$$\phi(t) = (c + t^2)^{1/2}, \qquad t \in \mathbb{R}^+,$$

655 Avenue of the Americas, New York, NY 10010 0024-3795/94/$7.00

where c is a positive constant, occurs in the "multiquadric" interpolation method of Hardy [8], which is highly successful in practice; see Dyn [5], Franke [6, 7], Kansa and Carlson [9], and the references therein. The existence of an interpolant for arbitrary given data depends upon the invertibility of the interpolation matrix A whose elements are

$$A_{jk} = \phi(|x_j - x_k|).$$

Micchelli [10] gave several classes of functions for which the interpolation matrices are nonsingular. In particular, he showed that the interpolation matrix associated with the multiquadrics is nonsingular, settling a conjecture of Franke [7]. We note that the nonsingularity of this interpolation matrix can also be derived from a result of Madych and Nelson [11, 12] concerning conditionally positive definite functions.

In many practical problems, it is often desirable to interpolate not only the function values but also the values of derivatives up to certain order, as in the classical Hermite interpolation on $\mathbb{R}$; see Davis [4]. To the benefit of application, we formulate the problem in the following general setting.

Let $\mathscr{P} := \{p_1, \ldots, p_r\}$ be a set of r linearly independent homogeneous polynomials on $\mathbb{R}^d$ with complex coefficients, and let $p_\mu(D)$ $(\mu = 1, \ldots, r)$ be the corresponding differential operators. Let $h_1, \ldots, h_r$ be complex-valued continuous functions on $\mathbb{R}^d$ such that the functions $p_\mu(D)h_\nu(\mu, \nu = 1, \ldots, r)$ are also continuous on $\mathbb{R}^d$. A vector-valued data function $\Delta : \mathscr{N} \mapsto \mathbb{C}^r$ is given, and we seek a function f in the linear space generated by the N $(N = nr)$ functions $h_\nu(\cdot - x_j)$ $(\nu = 1, \ldots, r,\ j = 1, \ldots, n)$ such that the vector-valued function $F : \mathbb{R}^d \mapsto \mathbb{C}^r$ defined by

$$F := (p_1(D)f, \ldots, p_r(D)f)$$

interpolates the given data, i.e., $F|\mathscr{N} = \Delta$.

We refer to this as the *scattered Hermite interpolation problem* (SHIP). When the interpolation conditions are imposed, the result is an $N \times N$ matrix A given in blocks by $A = (A_{\mu\nu})_{\mu,\nu=1}^r$, where $A_{\mu\nu}$ denotes the $n \times n$ matrix

$$\big((p_\mu(D)h_\nu)(x_j - x_k)\big)_{j,k=1}^n.$$

We will call A the SHIP matrix associated with $\mathscr{P}$ and the functions $h_1, \ldots, h_r$, or simply the SHIP matrix if no confusion is likely to occur. In order that the SHIP be uniquely solvable it is necessary and sufficient that the SHIP matrix be nonsingular.

In this paper, we study scattered Hermite interpolation by using radial basis functions. We will find several classes of functions that may be implemented for this interpolation scheme. These functions are closely related to those described in [10] and include the celebrated multiquadrics.

2. BASIC RESULT AND EXAMPLES

Let β be a finite Borel measure on $\mathbb{R}^d$. The Fourier transform $\hat{\beta}$ of β is the function defined by

$$\hat{\beta}(x) = \int_{\mathbb{R}^d} e^{ix\xi}\, d\beta(\xi).$$

It is obvious that the function $\hat{\beta}$ is uniformly continuous on $\mathbb{R}^d$. Let $M_{\mathscr{P}}$ denote the set of all positive finite Borel measures β on $\mathbb{R}^d$ that satisfy the following conditions:

(1) $\int_{\mathbb{R}^d} |x|^{2\gamma}\, d\beta(\xi) < \infty$, where $\gamma = \max_{1 \leqslant \nu \leqslant r}\{\deg p_\nu\}$.
(2) β is not concentrated on a subset of $\mathbb{R}^d$ with Lebesgue measure zero.

We will refer to the above two conditions as *measure conditions*. Let $\hat{M}_{\mathscr{P}} = \{\hat{\beta} : \beta \in M_{\mathscr{P}}\}$, and let $f \in \hat{M}_{\mathscr{P}}$. Then f possesses the following properties:

(1) The functions $p_\mu(D)(\bar{p}_\nu(D)f)$ ($\mu, \nu = 1, \dots, r$) are uniformly continuous on $\mathbb{R}^d$, and there is a positive finite Borel measure on $\mathbb{R}^d$ that satisfies the measure conditions such that the following relations hold:

$$\big(p_\mu(D)(\bar{p}_\nu(D)f)\big)(x) = \int_{\mathbb{R}^d} p_\mu(i\xi)\bar{p}_\nu(i\xi)e^{ix\xi}\, d\beta(\xi)$$
$$(\mu, \nu = 1, \dots, r),$$

where $\bar{p}_\mu$ is the polynomial whose coefficients are conjugates of those of p_μ.
(2) f is positive definite; see [14, p. 290].

The reason that we are interested in the functions in $\hat{M}_{\mathscr{P}}$ lies in the following theorem.

THEOREM 2.1. *Let $f \in \hat{M}_{\mathscr{P}}$, and set $h_\nu = (-1)^{\gamma_\nu}\bar{p}_\nu(D)f$ ($v = 1, \dots, r$), where $\gamma_\nu = \deg p_\nu$. Then the SHIP matrix associated with $\mathscr{P}$ and the functions $h_1, \dots, h_r$ is positive definite.*

Proof. We recall that the SHIP matrix A in question is given in blocks by $A = (A_{\mu\nu})_{\mu,\nu=1}^{r}$, where $A_{\mu\nu}$ is the $n \times n$ matrix

$$\left(\left[(-1)^{\gamma_\nu} p_\mu(D)(\bar{p}_\nu(D)f)\right](x_j - x_k)\right)_{j,k=1}^{n}.$$

To show that A is positive definite, let $C \in \mathbb{C}^N$ and $C \neq 0$. We write

$$C = (c_{11}, \ldots, c_{1n}, c_{21}, \ldots, c_{2n}, \ldots, c_{r1}, \ldots, c_{rn}),$$

and use $\bar{C}$ to denote the vector whose components are conjugates of those of C. We have

$$\begin{aligned}
C^T A \bar{C} &= \sum_{\mu=1}^{r} \sum_{j=1}^{n} \sum_{\nu=1}^{r} \sum_{k=1}^{n} \left[(-1)^{\gamma_\nu} p_\mu(D)(\bar{p}_\nu(D)f)\right](x_j - x_k) c_{\mu j} \bar{c}_{\nu k} \\
&= \sum_{\mu=1}^{r} \sum_{j=1}^{n} \sum_{\nu=1}^{r} \sum_{k=1}^{n} c_{\mu j} \bar{c}_{\nu k} \int_{\mathbb{R}^d} (-1)^{\gamma_\nu} p_\mu(i\xi) \bar{p}_\nu(i\xi) e^{i\xi(x_j - x_k)} d\beta(\xi) \\
&= \int_{\mathbb{R}^d} \sum_{\mu=1}^{r} \sum_{\nu=1}^{r} (-1)^{\gamma_\nu} p_\mu(i\xi) \bar{p}_\nu(i\xi) \sum_{j=1}^{n} \sum_{k=1}^{n} c_{\mu j} \bar{c}_{\nu k} e^{i\xi(x_j - x_k)} \, d\beta(\xi) \\
&= \int_{\mathbb{R}^d} \sum_{\mu=1}^{r} \sum_{\nu=1}^{r} (-1)^{\gamma_\nu} p_\mu(i\xi) \bar{p}_\nu(i\xi) \\
&\quad \times \left(\sum_{j=1}^{n} c_{\mu j} e^{i\xi x_j} \right) \left(\overline{\sum_{j=1}^{n} c_{\nu j} e^{i\xi x_j}} \right) d\beta(\xi) \\
&= \int_{\mathbb{R}^d} \left| \sum_{\mu=1}^{r} p_\mu(i\xi) g_\mu(\xi) \right|^2 d\beta(\xi), \qquad (2.1)
\end{aligned}$$

where g_μ denotes the function $\xi \mapsto \sum_{j=1}^{r} c_{\mu j} e^{i\xi x_j}$. Since $x_1, \ldots, x_n$ are distinct and $p_1, \ldots, p_r$ are linearly independent, the function

$$\xi \mapsto \sum_{\mu=1}^{r} p_\mu(i\xi) g_\mu(\xi)$$

is a nontrivial analytic function of ξ. Therefore it can only vanish on a subset of $\mathbb{R}^d$ with Lebesgue measure zero. Since the measure β is not concentrated on a subset of $\mathbb{R}^d$ with Lebesgue measure zero, it follows from (2.1) that $C^T A \bar{C} > 0$. ∎

The set $\hat{M}_{\mathscr{P}}$ is rich in functions. Our first example concerns the *box splines*, which have been extensively studied in the literature. We refer to Chui [3] for their basic properties. Let B be a box spline with direction set Ξ that spans $\mathbb{R}^d$, and let X be the subset of Ξ that satisfies the following conditions:

(1) $X = \{\xi_{11}, \dots, \xi_{1d}, \xi_{21}, \dots, \xi_{2d}, \dots, \xi_{l1}, \dots, \xi_{ld}\}$, where for each ν $(1 \leqslant \nu \leqslant l)$, the d vectors $\xi_{\nu 1}, \dots, \xi_{\nu n}$ are linearly independent.

(2) The set $\Xi \setminus X$ does not span $\mathbb{R}^d$.

Then we say that Ξ spans $\mathbb{R}^d$ l times.

COROLLARY 2.2. *Let B be a box spline with direction set Ξ that spans $\mathbb{R}^d$ l times. If all the multiplicities of the directions are even and if $l > 2\gamma + d$, then $B \in \hat{M}_{\mathscr{P}}$.*

Proof. It is well known that B is the Fourier transform of the measure β given by

$$\beta(\xi) = \prod_{\xi_j \in \Xi} \frac{\sin(\xi\xi_j/2)}{\xi\xi_j/2}\, d\xi.$$

It is clear that β is not concentrated on a set of $\mathbb{R}^d$ with Lebesgue measure zero. If all the multiplicities of the direction are even, then β is a positive Borel measure on $\mathbb{R}^d$. If $l > 2\gamma + d$, then we have

$$\int_{\mathbb{R}^d} |\xi|^{2\gamma}\, d\beta(\xi) < \infty.$$

Therefore, the desired result follows. ∎

COROLLARY 2.3. *For any fixed $c > 0$ and $\delta < 0$, the function $\Phi_{c,\delta}$ defined by $\Phi_{c,\delta}(x) = (c^2 + |x|^2)^\delta$ belongs to $\hat{M}_{\mathscr{P}}$.*

Proof. Without loss of generality, we assume that $c = 1$. The function $\Phi_{1,\delta}$ is the Fourier transform of the measure β_δ given by

$$\beta_\delta(\xi) = \frac{K_{d/2-\delta}(|\xi|)}{2^{\delta-1}(2\pi)^{d/2}\Gamma(\delta)|\xi|^{d/2-\delta}}\, d\xi,$$

(see Micchelli [10]), where K_κ is often called Macdonald's function; see Watson [17, pp. 78–79]. Furthermore, the function K_κ is positive and infinitely differentiable on $\mathbb{R}^+ \setminus \{0\}$, and decays exponentially at infinity; see [1, p. 374]. It follows that β_δ satisfies the measure conditions. ∎

COROLLARY 2.4. *For any fixed $t > 0$, the function G_t defined by $G_t(x) = e^{-t|x|^2}$ belongs to $\hat{M}_{\mathscr{P}}$.*

Proof. It is well known that G_t is the Fourier transform of the measure β_t is given by

$$\beta_t(\xi) = (2\pi)^{-d/2} e^{-|\xi|^2/(2\sqrt{2t})}\, d\xi.$$

It is obvious that β_t satisfies the measure conditions. ■

3. MAIN RESULTS

In this section, we concentrate on the finding of such functions ϕ from $\mathbb{R}^+$ to $\mathbb{R}$ that the function Φ defined by $\Phi(x) = \phi(|x|^2)$ along with its derivatives can be employed for scattered Hermite interpolation in Euclidean spaces $\mathbb{R}^d$ for any $d = 1, 2, \ldots$. We will be interested in two classes of functions that are closely related to completely monotone functions. Recall that a function ϕ from $\mathbb{R}^+$ to $\mathbb{R}$ is said to be *completely monotone* on $(0, \infty)$ if

$$(-1)^k \phi^{(k)}(t) \geqslant 0 \qquad \text{for all } t > 0 \text{ and all } k = 0, 1, 2 \ldots.$$

Let θ be a nonnegative integer, and let $C^\theta([0, \infty))$ denote the family of all functions from $\mathbb{R}^+$ to $\mathbb{C}$ that have continuous derivatives of orders up to and including θ, where the values of the derivatives at 0 are interpreted as the one-sided derivatives. Let $\mathscr{C}\mathscr{M}_\theta$ denote the class of functions ϕ that satisfy the following conditions:

(1) $\phi \in C^\theta([0, \infty))$.
(2) ϕ is completely monotone on $(0, \infty)$.
(3) ϕ is not a constant.

Let $\mathscr{D}\mathscr{M}_\theta$ denote the class of functions ϕ that satisfy the following conditions:

(1) $\phi \in C^\theta([0, \infty))$ and $\phi(0) > 0$.[1]
(2) ϕ' is completely monotone on $(0, \infty)$.
(3) ϕ is not a constant.

[1] If we just assume that $\phi(0) \geqslant 0$, then Theorem 3.4 in this section is only true for $n \geqslant 2$. Recall that n is the number of nodes.

For any fixed $c > 0$ and $\delta < 1$, let $\phi_{c,\delta}$ denote the function defined by $\phi_{c,\delta}(t) = (c + t)^{\delta}$ $(t \geqslant 0)$. Then it is easy to verify that (1) $\phi_{c,\delta}$ is an element of $\mathscr{CM}_\theta$ for all $\theta = 0, 1, 2, \ldots$ if $\delta < 0$ and (2) $\phi_{c,\delta}$ is an element of $\mathscr{DM}_\theta$ for all $\theta = 0, 1, 2, \ldots$ if $0 < \delta < 1$. The function $\phi_{1,1/2}(|\cdot|)$ is often referred to as the *multiquadric*, and the function $\phi_{1,-1/2}(|\cdot|)$ as the *inverse multiquadric*.

Functions in $\mathscr{CM}_\theta$ and $\mathscr{DM}_\theta$ are characterized in the following lemmas.

LEMMA 3.1. *In order that ϕ be an element of $\mathscr{CM}_\theta$, it is necessary and sufficient that ϕ have the following Laplace integral representation*

$$\phi(s) = \int_0^\infty e^{-st}\, d\alpha(t), \qquad s \geqslant 0, \tag{3.1a}$$

where α is a nondecreasing finite-valued function on $[0, \infty)$ such that

$$\int_0^\infty t^\theta\, d\alpha(t) < \infty \quad \text{and} \quad \int_{0^+}^\infty d\alpha(t) > 0. \tag{3.1b}$$

LEMMA 3.2. *In order that ϕ be an element of $\mathscr{DM}_\theta$, it is necessary and sufficient that ϕ have the following Laplace integral representations:*

$$\phi(s) - \phi(0) = \int_0^\infty \frac{1 - e^{-st}}{t}\, d\alpha(t), \qquad s \geqslant 0, \tag{3.2a}$$

where α is a nondecreasing finite-valued function on $[0, \infty)$ that satisfies the following conditions:

$$\int_0^\infty t^{\theta - 1}\, d\alpha(t) < \infty, \tag{3.2b}$$

$$\int_\varepsilon^\infty t^{-1}\, d\alpha(t) < \infty \qquad \text{for any fixed } \varepsilon > 0, \tag{3.2c}$$

$$\int_{0^+}^\infty d\alpha(t) > 0. \tag{3.2d}$$

Both Lemma 3.1 and Lemma 3.2 are consequences of the well-known Bernstein theorem; see Widder [16, p. 160]. Since the proof of Lemma 3.1 is simpler than and similar to that of Lemma 3.2, we omit the former and elaborate on the latter.

Proof of Lemma 3.2. To prove the sufficiency, assume that ϕ has the integral representation of (3.2a) in which the measure α satisfies the conditions (3.2b)–(3.2d). For $s > 0$, the conditions in (3.2b), (3.2c) and the rapid decay of the function $t \mapsto e^{-st}$ at infinity allow us to differentiate under the integral sign to get

$$\phi^{(k)}(s) = (-1)^{(k+1)} \int_0^\infty t^{k-1} e^{-st}\, d\alpha(t), \qquad k = 1, 2, \ldots, \theta. \quad (3.3)$$

It follows that ϕ' is completely monotone on $(0, \infty)$. In (3.3), letting $s \downarrow 0$ and using the monotone convergence theorem and (3.2b), we have

$$\phi^{(k)}(0) = (-1)^{(k+1)} \int_0^\infty t^{k-1}\, d\alpha(t), \qquad k = 1, 2, \ldots, \theta.$$

This shows that $\phi \in C^{\theta}([0, \infty))$. Finally, the conclusion that ϕ is not a constant follows from (3.2d).

To prove the necessity, assume that $\phi \in \mathscr{D}\mathscr{M}_\theta$. Since ϕ' is completely monotone on $(0, \infty)$, the Bernstein theorem asserts that ϕ is representable as the following Laplace integral:

$$\phi'(s) = \int_0^\infty e^{-st}\, d\alpha(t), \qquad s > 0, \quad (3.4)$$

where α is a nondecreasing finite-valued functions on $[0, \infty)$. Since ϕ is not a constant, it follows that $\int_{0^+}^\infty d\alpha(t) > 0$. Since $\phi(s) - \phi(\varepsilon) = \int_\varepsilon^s \phi'(\tau)\, d\tau$ is true for any $\varepsilon > 0$ and since ϕ is continuous at 0, we have $\phi(s) - \phi(0) = \int_0^s \phi'(\tau)\, d\tau$. Integrating from 0 to s on both sides of (3.4) and using Tonelli's theorem (see [13, p. 309]) on the right-hand side, we get (3.2a). And this implies that (3.2c) is also true. Finally, Equation (3.2b) follows from the assumption that $\phi \in C^{\theta}([0, \infty))$ as in the proof of the sufficient part. ∎

We now state and prove our main results.

THEOREM 3.3. *Let $\phi \in \mathscr{C}\mathscr{M}_{2\gamma}$ and let Φ be the function from $\mathbb{R}^d$ to $\mathbb{R}$ defined by $\Phi(x) = \phi(|x|^2)$. Then the SHIP matrix A associated with $\mathscr{P}$ and* *the functions*

$$h_\nu = (-1)^{\gamma_\nu}\bar{p}_\nu(D)\Phi \qquad (\nu = 1,\ldots,r)$$

is positive definite for any n distinct points $x_1,\ldots,x_n$ in $\mathbb{R}^d$ $(d = 1, 2, \ldots)$.

Proof. By Lemma 3.1, we can express ϕ in the Laplace integral form as in (3.1a) in which the measure α satisfies the conditions in (3.1b) with $\theta = 2\gamma$. Thus, we write

$$\Phi(x) = \int_0^\infty e^{-t|x|^2}\, d\alpha(t),$$

and we can differentiate under the integral sign[2] to get

$$\begin{aligned}(-1)^{\gamma_\nu}&\big(p_\mu(D)(\bar{p}_\nu(D)\Phi)\big)(x)\\ &= \int_0^\infty (-1)^{\gamma_\nu}\big(p_\mu(D)(\bar{p}_\nu(D)G_t)\big)(x)\, d\alpha(t).\end{aligned}$$

Recall from Corollary 2.4 that for any fixed $t > 0$, the function G_t is defined by $G_t(x) = e^{-t|x|^2}$. Let $C \in \mathbb{C}^N$ $(N = nr)$, $C \neq 0$. We show that $C^TA\bar{C} > 0$. Let E_t denote the SHIP matrix associated with $\mathscr{P}$ and the functions $h_\nu = (-1)^{\gamma_\nu}\bar{p}_\nu(D)G_t$ $(\nu = 1,\ldots,r)$. It is clear that the function $t \mapsto C^TE_t\bar{C}$ is continuous on $[0, \infty)$, and by Corollary 2.4, it is also positive on $(0, \infty)$. Therefore, it follows from the assumption $\int_{0^+}^\infty d\alpha(t) > 0$ that

$$C^TA\bar{C} = \int_0^\infty C^TE_t\bar{C}\, d\alpha(t) > 0. \qquad \blacksquare$$

THEOREM 3.4. *Let $\phi \in \mathscr{DM}_{2\gamma}$, and let Φ be the function from $\mathbb{R}^d$ to $\mathbb{R}$ defined by $\Phi(x) = \phi(|x|^2)$. Then the SHIP matrix A associated with $\mathscr{P}$ and the functions*

$$h_\nu = (-1)^{\gamma_\nu}\bar{p}_\nu(D)\Phi \qquad (\nu = 1,\ldots,r)$$

is nonsingular for any n distinct points $x_1,\ldots,x_n$ in $\mathbb{R}^d$ $(d = 1, 2, \ldots)$.

---

[2] Here we use the condition that $\int_0^\infty t^{2\gamma}\, d\alpha(t) < \infty$.

Proof. By Lemma 3.2, we can express ϕ in the Laplace integral form as in (3.2a) in which the measure α satisfies the conditions in (3.2b)–(3.2d) with $\theta = 2\gamma$. Thus, we have

$$\Phi(x) = \int_0^\infty \frac{1 - e^{-t|x|^2}}{t} d\alpha(t).$$

Without loss of generality, we assume $\gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_r$. We then consider two separate cases $\gamma_1 \geqslant 1$ and $\gamma_1 = 0$. In the first case, we show that the matrix A is negative definite. The conditions the measure α satisfies allows us to differentiate under the integral sign to get

$$\begin{aligned}&(-1)^{\gamma_\nu}\big(p_\mu(D)(\bar{p}_\nu(D)\Phi)\big)(x)\\ &\qquad = -\int_0^\infty t^{-1}(-1)^{\gamma_\nu}\big(p_\mu(D)(\bar{p}_\nu(D)G_t)\big)(x)\, d\alpha(t).\end{aligned}$$

Let $C \in \mathbb{C}^N$ $(N = nr)$, $C \neq 0$. The function $t \mapsto t^{-1}C^T E_t \overline{C}$ is well defined and continuous on $[0, \infty)$.[3] By Corollary 2.4, it is also positive on $(0, \infty)$. Therefore, it follows from the assumption $\int_{0^+}^\infty d\alpha(t) > 0$ that

$$C^T A \overline{C} = -\int_0^\infty t^{-1} C^T E_t \overline{C}\, d\alpha(t) < 0.$$

In the second case, we show that the matrix A has exactly $N - 1$ negative eigenvalues and 1 positive eigenvalue. This, in particular, implies that A is nonsingular. We have

$$\begin{aligned}&(-1)^{\gamma_\nu} p_\mu(D)(\bar{p}_\nu(D)\Phi)(x)\\ &\qquad = \begin{cases} \displaystyle\int_0^\infty \frac{1 - e^{-t|x|^2}}{t} d\alpha(t) & \text{if } \mu = \nu = 1,\\ \displaystyle -\int_0^\infty t^{-1}(-1)^{\gamma_\nu} p_\mu(D)(\bar{p}_\nu(D) g_t)(x)\, d\alpha(t) & \text{otherwise.}\end{cases}\end{aligned}$$

[3] In fact, the factor t^{-1} is canceled in the differentiation process.

Let $C = (c_{11}, \dots, c_{1n}, \dots, c_{r1}, \dots, c_{rn}) \in \mathbb{C}^N$, $C \neq 0$, such that $\sum_{j=1}^{n} c_{1j} = 0$. For such C, we have

$$C^T A \overline{C} = -\int_0^\infty t^{-1} C E_t \overline{C} \, d\alpha(t) < 0.$$

Since the set of all such C with real components is an $(N-1)$-dimensional space, by the Courant-Fischer min-max theorem (see [2, p. 113]), the matrix A has at least $N-1$ negative eigenvalues. Let e denote the vector $(1, \dots, 1, 0, \dots, 0, \dots, 0, \dots, 0)$. Then we have

$$e^T A e = \sum_{j=1}^{n} \sum_{k=1}^{n} \phi\left(|x_j - x_k|^2\right) > 0,$$

by the assumption that[4] $\phi(0) > 0$ and that ϕ is increasing on $\mathbb{R}^+$. Hence, the matrix A has at least 1 positive eigenvalue. It follows that A has exactly $N-1$ negative eigenvalues and 1 positive eigenvalue. ∎

The author has been inspired by the pioneering work of Professors W. Madych and C. Micchelli and benefited furthermore by having several useful discussions with them. The author thanks them for all this assistance. The referee kindly pointed out some typographical errors in the initial version of the paper.

REFERENCES

1 M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Appl. Math. Ser. 55, National Bureau of Standards, 1966; Dover.

2 R. Bellman, *Introduction to Matrix Analysis*, McGraw-Hill, New York, 1960.

3 C. K. Chui, *Multivariate Splines*, CBMS-NSF Conf. Ser., Providence, 1988.

4 P. J. Davis, *Interpolation & Approximation*, Dover, New York, 1975.

5 N. Dyn, Interpolation and approximation by radial and related functions, in *Approximation Theory VI*, Vol. I (C. K. Chui, L. L. Schumaker and J. D. Ward, Eds.), Academic, New York, 1989, pp. 211–234.

6 R. Franke, Scattered data interpolation: Test of some methods, *Math. Comp.* 38:181–200 (1982).

[4] We note that this is the only place where we use the assumption $\phi(0) > 0$. If we just assume that $\phi(0) \geqslant 0$, then the same argument proves the result for $n \geqslant 2$.

7 R. Franke, Recent advances in the approximation of surfaces from scattered data, in *Topics in Multivariate Interpolation* (C. K. Chui, L. L. Schumaker, and F. I. Utreras, Eds.), Academic, New York, 1987, pp. 79–98.
8 R. L. Hardy, Multiquadric equations of topography and other irregular surfaces, *J. Geophysical Rep.* 76:1905–1915 (1971).
9 E. J. Kansa and R. E. Carlson, Improved Accuracy of Multiquadric Interpolation Using Variable Shape Parameters, preprint, UCRL-JC-108658, 1991.
10 C. M. Micchelli, Interpolation of scattered data: Distance matrices and conditionally positive definite functions, *Constr. Approx.* 2:11–22 (1986).
11 W. Madych and S. Nelson, Multivariate interpolation and conditionally positive definite functions, *Approx. Theory Appl.* 4:77–89 (1988).
12 W. Madych and S. Nelson, Multivariate interpolation and conditionally positive definite functions II, *Math. Comp.* 54:211–230 (1990).
13 H. L. Royden, *Real Analysis*, 3rd ed., Macmillan, New York, 1988.
14 W. Rudin, *Functional Analysis*, McGraw-Hill, New York, 1973.
15 L. L. Schumaker, Fitting surfaces to scattered data, in *Approximation Theory II* (G. G. Lorentz, C. K. Chui, and L. L. Schumaker, Eds.), Academic, New York, 1976, pp. 203–268.
16 D. V. Widder, *The Laplace Transform*, Princeton U.P., Princeton, 1966.
17 G. M. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge U.P., 1962.

Received 21 October 1992; final manuscript accepted 26 February 1993